The Independent Domination Polynomial of Lollipop Graphs and Barbell Graph

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ABSTRACT

Let $G = (V, E)$ be a graph of order n. The independent domination polynomial of G is the polynomial $D_i(G,x) = \sum_{j=\gamma_i(G)}^{n} d_i(G,j)x^j$, where $d_i(G,j)$ is the number of independent dominating sets of G of size j. In this paper, we study the independent domination polynomial of a graph. The independent domination polynomial of Barbell graph and some Lollipop graphs are obtained.

Keywords: *Domination polynomial, independent domination polynomial.*
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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected connected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic graph theoretic terminology, we refer to Chartrand [2]. An independent set in a graph G is a set of pairwise non-adjacent vertices. A maximum independent set in G is a largest independent set and its size is called independence number of G and is denoted by $\alpha(G)$. A non-empty set $S \subseteq V(G)$ is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S and the minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$.

Independent domination problem is one of the interest in graph theory. The theory of independent domination was formalized by Berge [1] and Ore [5] in 1962 and the independent domination number was introduced by Cockayne and Hedetniemi[3]. Graph polynomial is one of the algebraic representations for graph. Saeid Alikhani and Peng, Y.H. [6] have introduced the domination polynomial of a graph. The independent domination polynomial in graphs introduced by P. M. Shivaswamy, N. D. Soner and Anwar Alwardi [7].

Theorem 1.1 A graph G of order n has domination number 1 if and only if G contains a vertex v of degree $n - 1$.

2. The independent domination polynomial of Barbell graph and Lollipop graphs

Definition 2.1. The Barbell graph is the graph obtained by connecting two copies of complete graph by a bridge and it is denoted by B_n.

Definition 2.2. The Lollipop graph $L_{n,m}$ for $n \geq 3$ as a graph obtained by joining complete graph K_n to a path P_m with a bridge.

Theorem 2.3. Let $G = B_n$ be a Barbell graph of order 2n, the independent domination polynomial of B_n is $D_i(B_n, x) = (n^2 - 1)x^2$.

Proof. Let $G = B_n$ be a Barbell graph of order 2n and let

$$V(G) = \{v_1, v_2, \dots, v_{n-1}, v_n, v_{n+1}, v_{n+2}, \dots, v_{2n}\}.$$

Let us take $A = \{v_1, v_2, \dots, v_{n-1}, v_n\}$ and $B = \{v_{n+1}, v_{n+2}, \dots, v_{2n}\}$, where A and B are the vertices of the two copies of K_n. We observe that there are two vertices which can dominate all the remaining vertices of G, it follows that $\gamma_i(B_n) \le 2$. To verify that $\gamma_i(B_n) \ge 2$, it is necessary to show that there is no dominating set with one vertex in G. There is no vertex of degree $2n-1$, and so by Theorem 1.1 $\gamma_i(B_n) \ge 2$. Therefore, $\gamma_i(B_n) = 2$.

There are $(n^2 - 1)$ independent dominating sets of cardinality 2. This can be obtained by choosing a vertex v_n and one vertex from B other than v_{n+1}; choosing the vertex v_{n+1} and one vertex from A other than v_n; choosing one vertex from A other than v_n and one vertex from B other than v_{n+1}. Therefore, $d_i(B_n, 2) = (n^2 - 1)$. Also there are no other ways to find the independent dominating sets of cardinality more than 2. Hence $D_i(B_n, x) = (n^2 - 1)x^2$.

Theorem 2.4. Let $G \cong L_{n,1}$ be a Lollipop graph with $(n+1)$ vertices. Then $D_i(G, x) = x + (n-1)x^2$.

Proof. Given $G \cong L_{n,1}$ is a Lollipop graph with $(n+1)$ vertices say $V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}\}$, where the vertices $v_i, 1 \le i \le n-1$, is of degree $(n-1)$; v_n is of degree n and v_{n+1} is of degree 1. Since the vertex v_n is of degree n, therefore by Theorem 1.1, $\gamma_i(G) = 1$ and so $d_i(G, 1) = 1$. Now choose a vertex v_{n+1} and one vertex from $V(G)$ except a vertex v_n. This produces independent dominating sets of cardinality 2. This can be done in $(n-1)$ ways. Therefore, $d_i(G, 2) = n - 1$. There are no other ways to find the independent dominating sets of cardinality 3or more. Hence $D_i(G, x) = x + (n-1)x^2$.

Theorem 2.5. For any Lollipop graph $L_{n,2}$, the independent domination polynomial is $D_i(L_{n,2}, x) = (2n-1)x^2$.

Proof. Let $G \cong L_{n,2}$ be a Lollipop graph with $(n+2)$ vertices. We take the vertex set as $V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}\}$ where the vertices $v_i, 1 \le i \le n-1$ is of degree $(n-1)$; v_n is of degree n;v_{n+1} is of degree 2 and v_{n+2} is of degree 1.

Clearly the set $\{v_i, v_{n+2}\}$ $\{v_i, v_{i+1}\}, \le i \le n$ gives a dominating set of cardinality 2, and so $\gamma_i(G) \le 2$. There is no vertex of degree $n+1$, therefore by Theorem 1.1, $\gamma_i(G) \ge 2$. Thus $\gamma_i(G) = 2$. Now choose a vertex v_{n+1} and one vertex from the complete graph K_n except the vertex v_n. Similarly choose a vertex v_{n+2} and one vertex from the complete graph K_n except the vertex v_n. Thus we get $(2n-1)$ independent dominating sets of cardinality 2. Therefore, $d_i(G, 2) = 2n - 1$. Clearly there are no other independent dominating sets of cardinality greater than two. Hence $D_i(G, x) = (2n-1)x^2$.

Theorem 2.6. Let $G \cong L_{n,3}$ be a Lollipop graph with $(n+3)$ vertices.Then the independent domination polynomial of G is $D_i(G, x) = (n+1)x^2 + (n-1)x^3$.

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Proof. Let $G \cong L_{n,3}$ be a Lollipop graph with $(n+3)$ vertices say $V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, v_{n+3}\}$ where the vertices $v_i, 1 \le i \le n-1$ of degree $(n-1)$; v_n is of degree $(n+1)$; the vertices v_{n+1} and v_{n+2} are of degree two and the vertex v_{n+3} is of degree one. Obviously, there are two vertices which dominate all the remaining vertices. Thus $\gamma_i(G) = 2$. Then the sets $\{v_n, v_{n+2}\}, \{v_n, v_{n+3}\}, \{v_1, v_{n+2}\}, \{v_2, v_{n+2}\}, \{v_3, v_{n+2}\}, \dots, \{v_{n-1}, v_{n+2}\}$ are the independent dominating sets of cardinality 2. Therefore, $d_i(G,2) = n+1$.

By definition, Let us assume that $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $V(P_3) = \{v_{n+1}, v_{n+2}, v_{n+3}\}$

Now, we choose a vertex $v_i, 1 \le i \le n-1$ from $V(K_n)$ and two vertices v_{n+1} *and* v_{n+3} from $V(P_3)$. This produces the independent dominating sets of cardinality 3.

That is the sets $\{v_1, v_{n+1}, v_{n+3}\}, \{v_2, v_{n+1}, v_{n+3}\}, \dots, \{v_{n-1}, v_{n+1}, v_{n+3}\}$.

Therefore, $d_i(G,3) = n-1$. Finally, it is impossible to find the other independent dominating sets of cardinality more than 3. Hence $D_i(G,x) = (n+1)x^2 + (n-1)x^3$

Theorem 2.7. Let $G \cong L_{n,4}$ be a Lollipop graph with $(n+4)$ vertices.Then the independent domination polynomial of G is $D_i(G,x) = x + (3n-2)x^3$.

Proof. Given $G \cong L_{n,4}$ a Lollipop graph. Label the vertices of G as $V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}\}$ where the vertices $v_i, 1 \le i \le n-1$ of degree $(n-1)$; v_n is of degree $(n+1)$; the vertices $v_{n+1}, v_{n+2},$ *and* v_{n+3} are of degree two and the vertex v_{n+4} is of degree one.

By definition, let us take $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $V(P_4) = \{v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}\}$.

Clearly, the vertex set $\{v_n, v_{n+3}\}$ dominate all the remaining vertices.

Therefore, $\gamma_i(G) = 2$. Thus, $d_i(G,1) = 1$

Now, we choose one vertex $v_i(1 \le i \le n-1)$ from $V(K_n)$ and choose two vertices from $V(P_4)$ which are of degree 2; choose one vertex $v_i(1 \le i \le n-1)$, one vertex is of degree 2 and one vertex of is degree one from $V(P_4)$; choose a vertex v_n, one vertex of degree 2 and a vertex of degree 1 from $V(P_4)$. This produces the independent dominating sets of cardinality 3.

That is the sets $\{v_1, v_{n+1}, v_{n+3}\}, \{v_2, v_{n+1}, v_{n+3}\}, \dots, \{v_{n-1}, v_{n+1}, v_{n+3}\}, \{v_1, v_{n+2}, v_{n+4}\}, \{v_2, v_{n+2}, v_{n+4}\}, \dots, \{v_{n-1}, v_{n+2}, v_{n+4}\}, \{v_n, v_{n+2}, v_{n+4}\}$are of cardinality 3.

Therefore, $d_i(G,3) = 3n-2$. Clearly, there are no other independent dominating sets of cardinality 4 and more. Hence $D_i(G,x) = x + (3n-2)x^3$.

Theorem 2.8. For any Lollipop graph $L_{n,5}$, the independent domination polynomial is $D_i(L_{n,5}, x) = 3nx^3 + (n-1)x^4$.

Proof. Let $G \cong L_{n,5}$ be a Lollipop graph with $(n+5)$ vertices. We take

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$V(G) = \{v_1, v_2, \dots, v_n, v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, v_{n+5}\}$ where the vertices v_i, $1 \le i \le n-1$ [are of deg]ree $(n-1)$; v_n is of degree $(n+1)$; the vertices $v_{n+1}, v_{n+2}, v_{n+3}$ and v_{n+4}, are of degree 2 and the [ver]tex v_{n+5} is of degree 1.

Let us assume that $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and $V(P_5) = \{v_{n+1}, v_{n+2}, v_{n+3}, v_{n+4}, v_{n+5}\}$

Obviously, there are three vertices which dominate all the remaining vertices. Therefore, $\gamma_i(G) = 3$

Then the independent dominating sets of cardinality 3 can be obtained by choosing a vertex v_n, one vertex from $V(P_5)$ is of degree 2 and the vertex of degree 1 from $V(P_5)$; choose one vertex from $V(K_n)$ except a vertex v_n and two vertices from $V(P_5)$ of degree 2; choose a vertex v_i, $1 \le i \le n-1$ from $V(K_n)$, one vertex from $V(P_5)$ of degree 2 and one vertex from $V(K_n)$ of degree 1. Therefore, $d_i(G, 3) = 3n$.

Also, the sets $\{v_1, v_{n+1}, v_{n+3}, v_{n+5}\}, \{v_2, v_{n+1}, v_{n+3}, v_{n+5}\}, \dots, \{v_{n-1}, v_{n+1}, v_{n+3}, v_{n+5}\}$ are the independent dominating sets of cardinality 4. Thus, $d_i(G, 4) = n - 1$

Clearly, there are no other ways to find the other independent dominating sets of cardinality more than four. Hence $D_i(L_{n,5}, x) = 3nx^3 + (n-1)x^4$.

Conclusion

The independent dominating polynomial of a graph is one of the algebraic representation of the graph and quality of any graph representation depend about what information can we get from that presentation about the graph.

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